THE KALMAN FILTER:
A LOOK BEHIND THE SCENE

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Presented at the Victorian Regional Survey Conference, Mildura, 23-25 June, 2006

ABSTRACT

The *Kalman filter* is a set of equations, applied recursively, that can be used in surveying applications to obtain position, velocity and acceleration of a moving object from traditional surveying measurements. The aim of this paper is to introduce the surveyor to the Kalman filter by examination of two simple applications, (i) Electromagnetic Distance Measurement (EDM) and (ii) the position and velocity of a ship in a navigation channel.

INTRODUCTION

A Kalman filter is a set of equations that are applied recursively to estimate the *state* of a *system* from a sequence of *noisy* measurements at times $t_1, t_2, t_3, \ldots$ etc. The "state" of the system is its value or values at times $t_1, t_2, t_3, \ldots$ etc and a "system" may have a single value or multiple values. Say, for instance, the system is a ship steaming on a particular heading in a shipping channel and the state of the system (the ship) is its east and north coordinates (E_k, N_k) and its velocity $(\dot{E}_k, \dot{N}_k)$. We say that this system (the ship) has a *state vector*[1] $\mathbf{x}_k = \left[E_k, N_k, \dot{E}_k, \dot{N}_k\right]^T$ containing four elements and the subscript k indicates a value at time t_k.

On the other hand, a system may be a process such as EDM by phase comparison of emitted and reflected light beams. The state of this system is a single value, the distance (D_k), determined at times $t_1, t_2, t_3, \ldots$ etc, and this system (the EDM) has a state vector $\mathbf{x}_k = [D_k]$ containing a single element and the subscript k indicates a value at time t_k.

"Noisy" measurements are measurements that contain small *random errors* assumed to be *normally distributed*, i.e., the aggregation of errors in size groupings would form the familiar symmetric bell-shaped histogram with positive and negative errors equally likely and small errors more frequent than large errors. Surveyors usually talk of residuals (or corrections) rather than errors, where a residual is the same magnitude as an error but of opposite sign.

A Kalman filter gives the best estimates (in a least squares sense) of the state of a *dynamic system* at a particular instant of time. And a dynamic system can be one whose values are changing with time, due to the motion of the system and measurement errors, or one whose values are measured at various instants of time and appear to change due to measurement errors. Dynamic systems do not have a single state (consisting of one or many values) that can be determined from a finite set of measurements but instead have a continuously changing state that has values sampled at different instants of time.

This paper aims to provide some insight into the Kalman filter and its implementation by studying two examples (i) the determination of a theoretical distance by an EDM; a dynamic system with a state vector containing a single value, and (ii) the determination of the position and velocity of a ship in a navigation channel; a dynamic system having a state vector containing four parameters.

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[1] In this paper, vectors are taken to mean *column-vectors*. A *row-vector* containing one row and n columns, and shown as $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ is the transpose of the column vector containing n rows and one column. To save space, column-vectors are often shown as $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ where the notation $[\;]^T$ indicates the transpose.
A vector can also contain a single element.

THE KALMAN FILTER

The Kalman filter equations were published in 1960 by Dr. R.E. Kalman in his famous paper describing a new approach to the solution of linear filtering and prediction (Kalman 1960). Since that time, papers on the application of the technique have been filling numerous scientific journals and it is regarded as one of the most important algorithmic techniques ever devised. It has been used in applications ranging from navigating the Ranger and Apollo spacecraft in their lunar missions to predicting short-term fluctuations in the stock market. Sorenson (1970) shows Kalman's technique to be an extension of C.F. Gauss' original method of least squares developed in 1795 and provides an historical commentary on its practical solution of linear filtering problems studied by 20th century mathematicians.

The derivation of the Kalman filter equations can be found in many textbooks related to signal processing that is the usual domain of Electrical Engineers, e.g., Brown and Hwang (1992). These derivations often use terminology that is unfamiliar to surveyors, but two authors, Krakiwsky (1975) and Cross (1992) both with geodesy and surveying backgrounds, have derivations, explanations, terminology and examples that would be familiar to any surveyor. This paper uses terminology similar to Cross and Krakiwsky. The Kalman filter equations and the associated measurement and dynamic models are given below with a brief explanation of the terms. It is hoped that the study of the two examples will help to make the Kalman filter a relatively easily understood process.

The *primary models* (or *measurement models*) at times t_{k-1} and t_k, and the *secondary model* (or *dynamic model*) linking the states at t_{k-1} and t_k are given by the matrix equations

$$\begin{aligned} \mathbf{v}_{k-1} + \mathbf{B}_{k-1}\mathbf{x}_{k-1} &= \mathbf{f}_{k-1} && \text{primary } t_{k-1} \\ \mathbf{v}_k + \mathbf{B}_k\mathbf{x}_k &= \mathbf{f}_k && \text{primary } t_k \qquad (1) \\ \mathbf{x}_k &= \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m && \text{dynamic} \end{aligned}$$

where

- $\mathbf{x}$ is the state vector
- $\mathbf{v}$ is the vector of residuals associated with the measurements
- $\mathbf{B}$ is a coefficient matrix
- $\mathbf{f}$ is a vector of numeric terms derived from the measurements
- $\mathbf{T}$ is the *transition* matrix
- $\mathbf{v}_m$ is a vector of residuals associated with the dynamic model.

Enforcing the least squares condition that the sum of the squares of the residuals, (multiplied by coefficients reflecting their precisions) be a minimum, gives rise to the set of recursive equations (the Kalman filter) that are applied as follows:

With initial estimates of the state vector $\mathbf{x}_{k-1}$ and the state cofactor matrix $\mathbf{Q}_{x_{k-1}}$ a Kalman filter has the following five general steps

(1) Compute the predicted state vector at t_k

$$\mathbf{x}'_k = \mathbf{T}\hat{\mathbf{x}}_{k-1} \qquad (2)$$

(2) Compute the predicted state cofactor matrix at t_k

$$\mathbf{Q}'_{x_k} = \mathbf{T}\mathbf{Q}_{x_{k-1}}\mathbf{T}^T + \mathbf{Q}_m \qquad (3)$$

(3) Compute the Kalman *Gain matrix*

$$\mathbf{K} = \mathbf{Q}'_{x_k}\mathbf{B}_k^T\left(\mathbf{Q} + \mathbf{B}_k\mathbf{Q}'_{x_k}\mathbf{B}_k^T\right)^{-1} \qquad (4)$$

(4) Compute the filtered state vector by updating the predicted state with the measurements at t_k

$$\hat{\mathbf{x}}_k = \mathbf{x}'_k + \mathbf{K}\left(\mathbf{f}_k - \mathbf{B}_k\mathbf{x}'_k\right) \qquad (5)$$

(5) Compute the filtered state cofactor matrix

$$\mathbf{Q}_{x_k} = \left(\mathbf{I} - \mathbf{K}_k\mathbf{B}_k\right)\mathbf{Q}'_{x_k} \qquad (6)$$

Go to step (1) and repeat the process for the next measurement epoch t_{k+1}.

The "hat" symbol $(\wedge)$ above the vector $\mathbf{x}$ indicates that it is an estimate of the true (but unknown) state of the system derived from the Kalman filter. This is also known as the filtered state. The "prime" symbol $(')$ indicates a predicted quantity. The superscript (*T*), e.g., $\mathbf{B}^T$ denotes the matrix transpose where the rows of $\mathbf{B}$ become the columns of $\mathbf{B}^T$. If $\mathbf{B}$ contains a single element, say $\mathbf{B} = [x]$ then $\mathbf{B}^T = \mathbf{B} = [x]$. The superscript (-1), e.g., $\mathbf{A}^{-1}$ denotes the inverse of a matrix and matrix inversion is defined by $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the diagonal Identity matrix (ones on the leading-diagonal). This relationship gives rise to matrix inversion routines that some computer languages offer as standard matrix functions.

It would be assumed that any computer implementation of a Kalman filter would use a language (C++, Visual Basic, etc.) that had some standard matrix routines attached that offered matrix transposition, multiplication and inversion. If a matrix has only a single element, say $\mathbf{A} = [x]$ then $\mathbf{A}^{-1} = [1/x]$.

Cofactor matrices, designated **Q** (often called covariance matrices) contain estimates of variances, denoted s_A^2, s_B^2, s_C^2, etc and covariances, denoted s_{AB}, s_{AC}, s_{BC}, etc associated with random quantities *A*, *B*, *C*, etc. The cofactor matrix of the measurements is denoted by **Q**, the cofactor matrix of the state vector is denoted by $\mathbf{Q}_x$ and the cofactor matrix of the dynamic model residuals is denoted by $\mathbf{Q}_m$. It should be noted that the cofactor matrix $\mathbf{Q}_m$ is derived in the following manner.

The dynamic model in equation (1)

$$\mathbf{x}_k = \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m \tag{7}$$

is an estimation of the true (but unknown) changes in the elements of the state vector from time t_{k-1} to time t_k and as such we assume that there are corrections to these estimations that are contained in the vector $\mathbf{v}_m$, the dynamic model residuals; and the elements of $\mathbf{v}_m$ are assumed to be small, random and normally distributed with a mean of zero. Also, we assume that the vector $\mathbf{v}_m$ is the product of two matrices, a coefficient matrix **H** and a vector **w** known as the *system driving noise*

$$\mathbf{v}_m = \mathbf{H}\mathbf{w} \tag{8}$$

The system driving noise **w** is a vector of random variables having variances and covariances contained in the cofactor matrix $\mathbf{Q}_w$ and applying the general law of propagation of variances to equation (8) gives

$$\mathbf{Q}_m = \mathbf{H}\mathbf{Q}_w\mathbf{H}^T \tag{9}$$

Determining **T**, **H**, **w**, $\mathbf{Q}_w$ and $\mathbf{Q}_m$ will be discussed in the examples below.

The Kalman filter equations are relatively easy to implement on modern computers (a reason for its popularity) and the examples studied below are supplemented by MATLAB[2] computer code available from the author.

DETERMINATION OF A DISTANCE BY AN EDM

The EDM component of a Total Station measures distances by phase comparison of an emitted and reflected modulated light beam. The measurement is an electro/optical process and the distance we see displayed after pressing the measure button on the Total Station is the "filtered" value of many hundreds of individual measurements, since a measurement takes only a number of milliseconds. This value could be the result of a Kalman filter process.

Consider the following sequence of measurements at times $t_1, t_2, t_3, \dots$ etc,

355.416, 355.430, 355.412, 355.402, 355.419, …

The variation in the measurements is assumed to be due to normally distributed random errors arising from the internal measurement process; often called the process noise. [The measurement sequence above, was generated by adding normally distributed random errors with mean zero and standard deviation 0.010 m to a constant value of 355.420 m.]

How will a Kalman filter produce the "filtered" value from this sequence?

First, let us assume that the measurement model is

$$\mathbf{l}_k + \mathbf{v}_k = \hat{\mathbf{l}}_k \tag{10}$$

$\mathbf{l}_k$ is the $(n \times 1)$ vector of measurements, $\mathbf{v}_k$ is the $(n \times 1)$ vector of residuals (small unknown corrections to the measurements) and $\hat{\mathbf{l}}_k$ are estimates of the true (but unknown) value of the measurements. n is the number of measurements at each epoch, that in this case is one. The primary measurement model can be expressed in terms of the filtered state vector $\hat{\mathbf{x}}_k$ at time t_k as

$$\mathbf{v}_k + \mathbf{B}_k\hat{\mathbf{x}}_k = \mathbf{f}_k \tag{11}$$

In this case $\hat{\mathbf{x}}_k$ contains the elements of $\hat{\mathbf{l}}_k$, both vectors containing single quantities and $\mathbf{f}_k = -\mathbf{l}_k$ also both containing single quantities (the measured distance at t_k). The matrix **B** will contain a single quantity, $\mathbf{B} = [-1]$.

[2] MATLAB, a registered trademark of The MathWorks, Inc., is a high-performance language for technical computing. It integrates computation, visualization, and programming in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation.

Secondly, the dynamic model linking the elements of the state vector at times t_{k-1} and t_k is

$$\mathbf{x}_k = \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m \tag{12}$$

The state vector contains a single element that should remain unchanged between t_{k-1} and t_k (any change is simply due to measurement errors) then the transition matrix $\mathbf{T}$ will contain a single element $\mathbf{T} = [1]$ and there are no assumed corrections to this model; hence $\mathbf{v}_m = \mathbf{0}$, $\mathbf{H} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$ and from equation (9) $\mathbf{Q}_m = \mathbf{0}$.

Lastly, an estimate of the cofactor matrix of the measurements $\mathbf{Q}$ and the elements of the state vector $\mathbf{Q}_{x_k}$ must be made. Let us assume (guess) that the measurements have a standard deviation of 10 mm (0.010 m) and hence their estimated variance is $(0.010)^2$ and $\mathbf{Q} = \left[(0.010)^2\right]$. Since our primary measurement model has a state vector containing a single value (the measurement), then $\mathbf{Q}_x$ will only contain a single value, and we have as a starting estimate $\mathbf{Q}_{x_1} = \left[(0.010)^2\right]$, the same as $\mathbf{Q}$.

Now we can now start the Kalman filter at epoch t_2 using the values at t_1 as filtered estimates.

(1) Compute the predicted state vector at epoch t_2 using the measurement 355.416 at t_1 as the filtered estimate $\hat{\mathbf{x}}_1$

$$\begin{aligned}\mathbf{x}'_2 &= \mathbf{T}\hat{\mathbf{x}}_1 \\ &= [1][355.416] \\ &= 355.416\end{aligned}$$

(2) Compute the predicted state cofactor matrix at t_2 using $\mathbf{Q}_{x_1} = \left[(0.010)^2\right]$ as the filtered estimate

$$\begin{aligned}\mathbf{Q}'_{x_2} &= \mathbf{T}\mathbf{Q}_{x_1}\mathbf{T}^T + \mathbf{Q}_m \\ &= [1]\left[(0.010)^2\right][1] + 0 \\ &= (0.010)^2\end{aligned}$$

(3) Compute the Kalman *Gain matrix* noting that $\mathbf{Q} = \left[(0.010)^2\right]$

$$\begin{aligned}\mathbf{K} &= \mathbf{Q}'_{x_2}\mathbf{B}_2^T\left(\mathbf{Q} + \mathbf{B}_2\mathbf{Q}'_{x_2}\mathbf{B}_2^T\right)^{-1} \\ &= \left[(0.010)^2\right][-1] \\ &\quad \times\left(\left[(0.010)^2\right] + [-1]\left[(0.010)^2\right][-1]\right)^{-1} \\ &= \left[-(0.010)^2\right]\left[2(0.010)^2\right]^{-1} \\ &= -0.500\end{aligned}$$

(4) Compute the filtered state vector $\hat{\mathbf{x}}_2$ by updating the predicted state with the measurements at t_2

$$\begin{aligned}\hat{\mathbf{x}}_2 &= \mathbf{x}'_2 + \mathbf{K}\left(\mathbf{f}_2 - \mathbf{B}_2\mathbf{x}'_2\right) \\ &= [355.416] + \{[-0.500] \\ &\quad \times([-355.430] - [-1][355.416])\} \\ &= 355.423\end{aligned}$$

(5) Compute the filtered state cofactor matrix at t_2

$$\begin{aligned}\mathbf{Q}_{x_2} &= \left(\mathbf{I} - \mathbf{K}_2\mathbf{B}_2\right)\mathbf{Q}'_{x_2} \\ &= \left([1] - [-0.500][-1]\right)\left[(0.010)^2\right] \\ &= 0.000050\end{aligned}$$

Go to step (1) and repeat the process for the next measurement epoch t_3.

The values from the Kalman filter for epochs t_3, t_4 and t_5 are

epoch t_3		epoch t_4	
$\mathbf{x}'_3$	$= 355.423000$	$\mathbf{x}'_4$	$= 355.419333\ldots$
$\mathbf{Q}'_{x_3}$	$= 0.000050$	$\mathbf{Q}'_{x_4}$	$= 0.0000333\ldots$
$\mathbf{K}_3$	$= -0.333333\ldots$	$\mathbf{K}_4$	$= -0.250000$
$\hat{\mathbf{x}}_3$	$= 355.419333\ldots$	$\hat{\mathbf{x}}_4$	$= 355.415000$
$\mathbf{Q}_{x_3}$	$= 0.000033\ldots$	$\mathbf{Q}_{x_4}$	$= 0.000025$

epoch t_5	
$\mathbf{x}'_5$	$= 355.415000$
$\mathbf{Q}'_{x_5}$	$= 0.000025$
$\mathbf{K}_5$	$= -0.200000$
$\hat{\mathbf{x}}_5$	$= 355.415800$
$\mathbf{Q}_{x_5}$	$= 0.000020$

So, the sequence of measurements is

355.416, 355.403, 355.421, 355.423, 355.408, …

and the Kalman filter estimates $\hat{\mathbf{x}}$ (the filtered values) are

355.416, 355.423, 355.419, 355.415, 355.416, …

Something that should be noted is that the filtered state cofactor matrix $\mathbf{Q}_x$ contains the estimate of the variance of the filtered value (variance is standard deviation squared). We started with an estimated value $\mathbf{Q}_{x_1} = \mathbf{Q} = (0.010)^2 = 0.000100$ and after five epochs the estimated value had reduced to $\mathbf{Q}_{x_5} = 0.000020$ equivalent to an estimated standard deviation of 0.0045 m.

So the Kalman filter gives estimates with a better precision than the assumed precision of the measurement sequence; as we should expect from a least squares process. A Kalman filter program *edm.m*, written in the MATLAB language and available from the author, processes 250 EDM measurements that are obtained by adding normally distributed random errors, with mean zero and standard deviation 0.010 m, to a constant value 355.420 m. Figure 1 below shows two plots from this program (i) the filtered estimate of the distance (the filtered state) as a black solid line and the 250 measurements as black dots and (ii) a plot of the standard deviation of the filtered distance. After processing the 250 measurements the filtered distance was 355.420 m with a standard deviation of 0.000632 m.

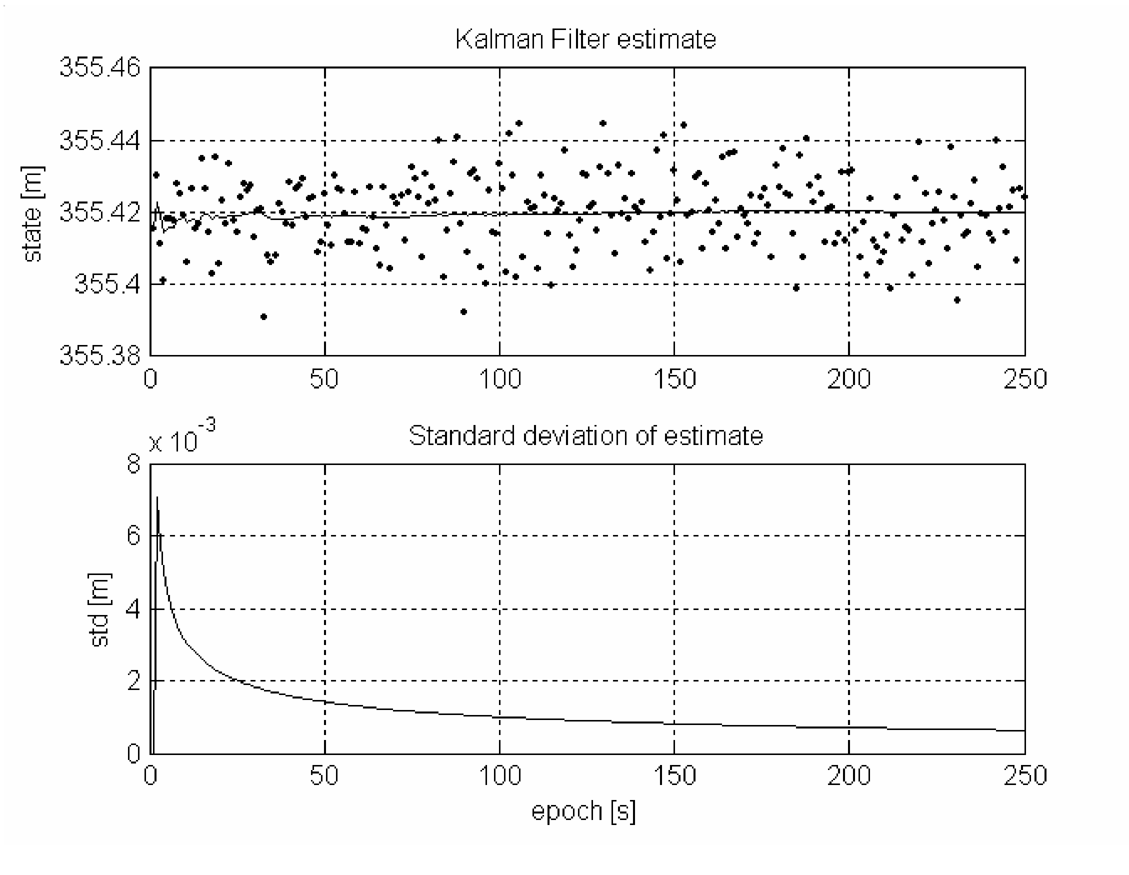

Figure 1. MATLAB plots of filtered state (EDM distance) and standard deviation of filtered state.

It is interesting to note that if all 250 measurements $x_1, x_2, x_3, \ldots, x_{250}$ (each with standard deviation $s_x = 0.010$ m) had been recorded and the mean $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_{250}}{250}$ computed, then propagation of variances gives the standard deviation of the mean as $s_{\bar{x}} = \dfrac{s_x}{\sqrt{250}} = 0.000632$ m which is the same as the Kalman filter result.

DETERMINATION OF POSITION AND VELOCITY OF A SHIP IN A NAVIGATION CHANNEL

Figure 2 shows the path of a ship as it moves down a shipping channel at a constant heading and speed. Navigation equipment on board automatically measures distances to transponders at three navigation beacons *A*, *B* and *C* at 60-second intervals. The measured distances are known to have a standard deviation of 1 metre and the solid line in Figure 2 represents solutions of the ship's position for each set of measurements at the 60-second time intervals. The true path of the ship is shown as the dashed line. The transponder measurements, from the ship to the navigation beacons *A*, *B* and *C* at 60-second time intervals are shown in Table 1 below. The data were generated by assuming the starting coordinates of the ship were 7875.000 m East and 6319.392 m North and the ship was travelling at 15 knots on a heading of 064º (1 knot = 1 nautical mile per hour and 1 nautical mile = 1852 metres). At 60-second intervals, the true ship position and distances to the beacons were computed. These distances were then "disturbed" by the addition of normally distributed random errors (with zero mean and standard deviation 1 metre) and then rounded to the nearest 0.1 m.

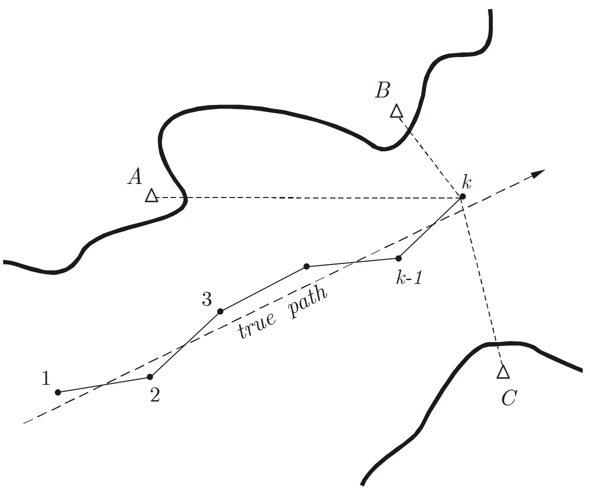

Figure 2. Path of a ship in a navigation channel. (*A*, *B* and *C* are known navigation beacons)

The coordinates of the three navigation beacons are:

A: 10000.000 E
10000.000 N

B: 13880.000 E
11250.000 N

C: 15550.000 E
7160.000 N

Epoch	Transponder measurements to navigation beacons		
	A	B	C
1	4249.7	7768.6	7721.1
2	3876.1	7321.4	7288.5
3	3518.4	6872.2	6857.6
4	3193.3	6426.0	6429.1
5	2903.6	5982.6	6009.7
6	2664.0	5543.2	5596.6
7	2490.9	5107.7	5191.5
8	2392.9	4678.9	4797.1
9	2383.2	4253.4	4417.8
10	2463.0	3841.7	4050.9
11	2623.2	3435.6	3709.9
12	2849.0	3054.2	3395.8
13	3126.7	2692.9	3119.4
14	3446.9	2366.6	2891.1
15	3793.4	2096.4	2724.4
16	4166.0	1900.6	2630.9
17	4552.2	1804.7	2610.2
18	4956.2	1824.8	2677.4
19	5366.4	1959.6	2819.7
20	5785.0	2182.8	3023.5

Table 1. Transponder measurements at 60-second time intervals

How will a Kalman filter produce an estimated position, speed and heading of the ship from the transponder measurements?

Note that in our Kalman filter, the state vector will be $\mathbf{x}_k = \left[E_k, N_k, \dot{E}_k, \dot{N}_k\right]^T$ containing $n = 4$ elements (or parameters) where $\left(E_k, N_k\right)$ are the ship's position and $\left(\dot{E}_k, \dot{N}_k\right)$ the ship's velocity components. The speed of the ship at time t_k is

$$speed = \sqrt{\left(\dot{E}_k\right)^2 + \left(\dot{N}_k\right)^2} \tag{13}$$

and the heading of the ship (bearing from North) at time t_k is

$$\tan\left(heading\right) = \frac{\dot{E}_k}{\dot{N}_k} \tag{14}$$

The measurement model (primary model)

Let us assume that the primary or measurement model is

$$\mathbf{l}_k + \mathbf{v}_k = \hat{\mathbf{l}}_k \tag{15}$$

where $\mathbf{l}_k$ is the $(m \times 1)$ vector of measurements (the transponder distances), $\mathbf{v}_k$ is the $(m \times 1)$ vector of residuals (small unknown corrections to the measurements) and $\hat{\mathbf{l}}_k$ are estimates of the true (but unknown) value of the measurements. m is the number of measurements, that in this case is three at each measurement epoch. The estimates $\hat{\mathbf{l}}_k$ are non-linear functions of the coordinates E,N of the beacons A, B and C and the filtered state coordinates $\hat{E}_k, \hat{N}_k$ of the ship at time t_k

$$\begin{aligned} \hat{l}_j &= \hat{l}\left(\hat{E}_k, \hat{N}_k, E_j, N_j\right) \\ &= \sqrt{\left(\hat{E}_k - E_j\right)^2 + \left(\hat{N}_k - N_j\right)^2} \end{aligned} \tag{16}$$

for $j = A, B, C$

Expanding equation (16) into a series using Taylor's theorem gives

$$\hat{l} = l' + \frac{\partial \hat{l}}{\partial \hat{E}_k}\left(\hat{E}_k - E'_k\right) + \frac{\partial \hat{l}}{\partial \hat{N}_k}\left(\hat{N}_k - N'_k\right) + \text{higher order terms}$$

where E'_k, N'_k are approximate coordinates of the ship at t_k, l' is an approximate distance computed using E'_k, N'_k and the coordinates of the beacon, and the partial derivatives are

$$\begin{aligned} \frac{\partial \hat{l}}{\partial \hat{E}_k} &= \frac{E'_k - E_j}{l'_j} = d_j \\ \frac{\partial \hat{l}}{\partial \hat{N}_k} &= \frac{N'_k - N_j}{l'_j} = c_j \quad \text{for} \quad j = A, B, C \end{aligned} \tag{17}$$

Re-arranging equation (15) for a single distance gives

$$v - \hat{l} = -l$$

and substituting the Taylor series approximation for $\hat{l}$ (ignoring higher-order terms) and re-arranging gives the linearized form of the primary measurement model as

$$v_j - d_j \hat{E}_k - c_j \hat{N}_k = l'_j - l_j + \left(-d_j E'_k - c_j N'_k\right) \tag{18}$$

for $j = A, B, C$

This primary measurement model can be expressed in terms of the filtered state vector $\hat{\mathbf{x}}_k$ at time t_k in the matrix form as

$$\begin{bmatrix} v_A \\ v_B \\ v_C \end{bmatrix}_k + \begin{bmatrix} -d_A & -c_A & 0 & 0 \\ -d_B & -c_B & 0 & 0 \\ -d_C & -c_C & 0 & 0 \end{bmatrix}_k \begin{bmatrix} \hat{E} \\ \hat{N} \\ \dot{E} \\ \dot{N} \end{bmatrix}_k$$

$$= \begin{bmatrix} l'_A - l_A \\ l'_B - l_B \\ l'_C - l_C \end{bmatrix}_k + \begin{bmatrix} -d_A & -c_A & 0 & 0 \\ -d_B & -c_B & 0 & 0 \\ -d_C & -c_C & 0 & 0 \end{bmatrix}_k \begin{bmatrix} E' \\ N' \\ \dot{E}' \\ \dot{N}' \end{bmatrix}_k$$

or $$\mathbf{v}_k + \mathbf{B}_k \hat{\mathbf{x}}_k = \mathbf{l}'_k - \mathbf{l}_k + \mathbf{B}_k \mathbf{x}'_k = \mathbf{f}_k \tag{19}$$

Now in step (4) of the Kalman filter algorithm [see equation (5)], the filtered state vector $\hat{\mathbf{x}}_k$ is obtained from

$$\hat{\mathbf{x}}_k = \mathbf{x}'_k + \mathbf{K}\left(\mathbf{f}_k - \mathbf{B}_k \mathbf{x}'_k\right) \tag{20}$$

and substituting for $\mathbf{f}_k$ from equation (19) gives

$$\begin{aligned} \hat{\mathbf{x}}_k &= \mathbf{x}'_k + \mathbf{K}\left(\mathbf{l}'_k - \mathbf{l}_k + \mathbf{B}_k \mathbf{x}'_k - \mathbf{B}_k \mathbf{x}'_k\right) \\ &= \mathbf{x}'_k + \mathbf{K}\left(\mathbf{l}'_k - \mathbf{l}_k\right) \end{aligned} \tag{21}$$

Note: (i) the term $\left(\mathbf{f}_k - \mathbf{B}_k \mathbf{x}'_k\right)$ in equation (20) is often called the *predicted residuals* $\mathbf{v}'_k$ where, in our case

$$\mathbf{v}'_k = \mathbf{f}_k - \mathbf{B}_k \mathbf{x}'_k = \mathbf{l}'_k - \mathbf{l}_k \tag{22}$$

(ii) The term $\mathbf{K}\left(\mathbf{l}'_k - \mathbf{l}_k\right)$ in equation (21) is often called the *corrections to the predicted state* $\Delta\mathbf{x}$ where, in our case

$$\Delta\mathbf{x}_k = \mathbf{K}_k\left(\mathbf{l}'_k - \mathbf{l}_k\right) \tag{23}$$

The dynamic model (secondary model)

A dynamic model that is extremely simple and often used in navigation problems can be developed by considering a continuous function of time, say $y = y(t)$. Following the development by Cross (1987), we can use Taylor's theorem to expand the function $y(t)$ about the point $t = t_k$ into the series

$$y(t) = y(t_k) + (t - t_k)\dot{y}(t_k) + \frac{(t - t_k)^2}{2!}\ddot{y}(t_k) + \frac{(t - t_k)^3}{3!}\dddot{y}(t_k) + \cdots$$

where $\dot{y}(t_k), \ddot{y}(t_k), \dddot{y}(t_k)$, etc are derivatives of y with respect to t evaluated at $t = t_k$. Letting $t = t_k + \Delta t$ and then $\Delta t = t - t_k$ we may write

$$y(t_k + \Delta t) = y(t_k) + \dot{y}(t_k)\Delta t + \frac{\ddot{y}(t_k)}{2!}(\Delta t)^2 + \frac{\dddot{y}(t_k)}{3!}(\Delta t)^3 + \cdots \tag{24}$$

We now have a power series expression for the continuous function $y(t)$ at the point $t = t_k + \Delta t$ involving the function y and its derivatives $\dot{y}, \ddot{y}$, etc, (all evaluated at t_k) and the time difference $\Delta t = t - t_k$.

In a similar manner, if we assume $\dot{y}(t), \ddot{y}(t)$, etc to be continuous functions of t, then

$$\begin{aligned}\dot{y}(t_k + \Delta t) &= \dot{y}(t_k) + \ddot{y}(t_k)\Delta t + \frac{\dddot{y}(t_k)}{2!}(\Delta t)^2 + \cdots \\ \ddot{y}(t_k + \Delta t) &= \ddot{y}(t_k) + \dddot{y}(t_k)\Delta t + \cdots \\ &\text{etc}\end{aligned} \tag{25}$$

Now consider two time epochs t_k and t_{k-1} separated by a time interval Δt, we can combine equations (24) and (25), with a change of subscripts for t, into the general matrix forms:

(i) involving terms up to $\ddot{y}$

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \left[\ddot{y}\right]_{k-1} \tag{26}$$

(ii) involving terms up to $\dddot{y}$

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}_k = \begin{bmatrix} 1 & \Delta t & \frac{1}{2}(\Delta t)^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{6}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \left[\dddot{y}\right]_{k-1} \tag{27}$$

In many navigation problems the continuous function of time $y = y(t)$ is simply *position*, so $y(t) = \{E, N\}(t)$ where E,N are east and north coordinates. Here $y(t) = \{\ \}(t)$ means y is a function of time t where the function contains the variables within the braces $\{\ \}$. The derivatives are *velocity:* $\dot{y}(t) = \{\dot{E}, \dot{N}\}(t)$, *acceleration:* $\ddot{y}(t) = \{\ddot{E}, \ddot{N}\}(t)$ and *jerk:* $\dddot{y}(t) = \{\dddot{E}, \dddot{N}\}(t)$ which is the rate of change of acceleration. In equations (26) and (27), we can consider the vector on the left-hand-side of the equals sign to be the vector $\mathbf{x}_k$, the state vector, or the state of the system at time t_k. The matrix on the right-hand-side is the *transition matrix* **T** and the elements of this matrix contain the links between the state vector at times t_k and t_{k-1}, i.e., $\mathbf{x}_k = \mathbf{T}\mathbf{x}_{k-1}$. The second term in the equations above is the product of two matrices and the result will be the vector of model residuals $\mathbf{v}_m$ (containing the same number of elements as the state vector). $\mathbf{v}_m$ is a reflection of the fact that the transition matrix does not fully describe the exact physical links between the states at times t_k and t_{k-1} and $\mathbf{v}_m = \mathbf{H}\mathbf{w}$ where **H** is a coefficient matrix and **w** is the *system driving noise*. In equations (26) and (27), the system driving noise is acceleration and jerk respectively.

We can now use these general forms to define a suitable dynamic model.

In our simple case (the ship in the channel) the state vector **x** contains four elements $\mathbf{x}_k = \left[E_k, N_k, \dot{E}_k, \dot{N}_k\right]^T$ and the appropriate dynamic model in the form of equation (26) is

$$\begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 & 0 \\ 0 & \frac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \ddot{E} \\ \ddot{N} \end{bmatrix}_{k-1} \tag{28}$$

or $\quad \mathbf{x}_k = \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m \qquad (29)$

where **T** is the $(n \times n)$ transition matrix and $\mathbf{v}_m$ is the $(n \times 1)$ vector of model residuals.

If we expand equation (28) we see that it is really just the matrix form of the two equations of rectilinear motion; (i) $v = u + at$ and (ii) $s = ut + \frac{1}{2}at^2$ where s is distance, u is initial velocity, v is final velocity, a is acceleration and t is time.

In our notation they are:

(i) $$\dot{E}_k = \dot{E}_{k-1} + \ddot{E}\,\Delta t \quad \text{and} \quad \dot{N}_k = \dot{N}_{k-1} + \ddot{N}\,\Delta t$$

(ii) $$E_k = E_{k-1} + \dot{E}_{k-1}\,\Delta t + \tfrac{1}{2}\ddot{E}\left(\Delta t\right)^2$$
$$N_k = N_{k-1} + \dot{N}_{k-1}\,\Delta t + \tfrac{1}{2}\ddot{N}\left(\Delta t\right)^2$$

The dynamic model residuals $\mathbf{v}_m = \mathbf{Hw}$ are

$$\begin{bmatrix} v_E \\ v_N \\ v_{\dot{E}} \\ v_{\dot{N}} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}(\Delta t)^2 & 0 \\ 0 & \tfrac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \ddot{E} \\ \ddot{N} \end{bmatrix} \tag{30}$$

where the coefficient matrix **H** and the system driving noise **w** are

$$\mathbf{H} = \begin{bmatrix} \tfrac{1}{2}(\Delta t)^2 & 0 \\ 0 & \tfrac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} \ddot{E} \\ \ddot{N} \end{bmatrix} \tag{31}$$

In this simple navigation problem it is assumed that the system driving noise **w** contains small random accelerations caused by the sea and wind conditions, the steering of the ship, the engine speed variation, etc.

The cofactor matrix of the dynamic model $\mathbf{Q}_m$ is given by

$$\mathbf{Q}_m = \mathbf{HQ}_w\mathbf{H}^T \tag{32}$$

where $\mathbf{Q}_w$, the cofactor matrix of the system driving noise, is

$$\mathbf{Q}_w = \begin{bmatrix} s^2_{\ddot{E}} & 0 \\ 0 & s^2_{\ddot{N}} \end{bmatrix}$$

and $s^2_{\ddot{E}}, s^2_{\ddot{N}}$ are the estimates of the variances of the accelerations in the east and north directions and the covariance is assumed to be zero. Using the coefficient matrix **H** in equation (31) we have

$$\mathbf{Q}_m = \begin{bmatrix} \tfrac{1}{2}(\Delta t)^2 & 0 \\ 0 & \tfrac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} s^2_{\ddot{E}} & 0 \\ 0 & s^2_{\ddot{N}} \end{bmatrix}$$
$$\times \begin{bmatrix} \tfrac{1}{2}(\Delta t)^2 & 0 & \Delta t & 0 \\ 0 & \tfrac{1}{2}(\Delta t)^2 & 0 & \Delta t \end{bmatrix} \tag{33}$$

Now we can start the Kalman filter, but first some initial values must be set. These initialisation steps will be designated as (a), (b), (c) etc followed by the Kalman filter steps (1), (2), (3) etc.

(a) Set the elements of the transition matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this exercise $\Delta t = 60\,\text{sec}$

(b) Set the cofactor matrix of the measurements

$$\mathbf{Q} = \begin{bmatrix} s^2_{l_A} & 0 & 0 \\ 0 & s^2_{l_B} & 0 \\ 0 & 0 & s^2_{l_C} \end{bmatrix}$$

In this case $s^2_{l_A} = s^2_{l_B} = s^2_{l_C} = 1.0\ \text{m}^2$

(c) Set the cofactor matrix of the system driving noise

$$\mathbf{Q}_w = \begin{bmatrix} s^2_{\ddot{E}} & 0 \\ 0 & s^2_{\ddot{N}} \end{bmatrix}$$

In this case $s^2_{\ddot{E}} = s^2_{\ddot{N}} = 0.017\ \text{m}^2/\text{s}^4$

(d) Set the coefficient matrix of the system driving noise

$$\mathbf{H} = \begin{bmatrix} \tfrac{1}{2}(\Delta t)^2 & 0 \\ 0 & \tfrac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}$$

(e) Compute the cofactor matrix of the dynamic model

$$\mathbf{Q}_m = \mathbf{HQ}_w\mathbf{H}^T$$

(f) Set the starting estimates of the state vector. This will be the filtered state vector for epoch t_2

$$\mathbf{x}_1 = \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_1 = \begin{bmatrix} 7875.000\ \text{m} \\ 6319.392\ \text{m} \\ 7\ \text{m}/s \\ 3\ \text{m/s} \end{bmatrix}$$

(g) Set the starting estimates of the state cofactor matrix. This will be the filtered state cofactor matrix for epoch t_2

$$\mathbf{Q}_{x_k} = \begin{bmatrix} s_E^2 & 0 & 0 & 0 \\ 0 & s_N^2 & 0 & 0 \\ 0 & 0 & s_{\dot{E}}^2 & 0 \\ 0 & 0 & 0 & s_{\dot{N}}^2 \end{bmatrix}$$

In this case $s_E^2 = s_N^2 = 20\,\mathrm{m}^2$ and $s_{\dot{E}}^2 = s_{\dot{N}}^2 = 0.5\ \mathrm{m}^2/\mathrm{s}^2$

Now start the Kalman filter at epoch t_2

(1) Compute the predicted state vector at epoch t_2 using the filtered estimate $\hat{\mathbf{x}}_1$

$$\mathbf{x}'_2 = \mathbf{T}\hat{\mathbf{x}}_1$$

(2) Compute the predicted state cofactor matrix at t_2 using $\mathbf{Q}_m$ from step (e)

$$\mathbf{Q}'_{x_2} = \mathbf{T}\mathbf{Q}_{x_1}\mathbf{T}^T + \mathbf{Q}_m$$

(3) Compute the Kalman *Gain matrix*

$$\mathbf{K} = \mathbf{Q}'_{x_2}\mathbf{B}_2^T\left(\mathbf{Q} + \mathbf{B}_2\mathbf{Q}'_{x_2}\mathbf{B}_2^T\right)^{-1}$$

Using **Q** from step (b) and **B** whose elements have been determined using equations (17). The form of **B** is given in equation (19).

(4.1) Compute the numeric terms "computed – observed" distances [see equation (21)]

(4.2) Compute the filtered state vector $\hat{\mathbf{x}}_2$ by updating the predicted state [see equation (21)]

$$\hat{\mathbf{x}}_k = \mathbf{x}'_k + \mathbf{K}\left(\mathbf{l}'_k - \mathbf{l}_k\right)$$

(5) Compute the filtered state cofactor matrix at t_2

$$\mathbf{Q}_{x_2} = \left(\mathbf{I} - \mathbf{K}_2\mathbf{B}_2\right)\mathbf{Q}'_{x_2}$$

Go to step (1) and repeat the process for the next measurement epoch t_3.

The following output from a MATLAB program *kalship3.m* (available from the author) processes the data in Table 1 in a Kalman filter beginning at epoch 2.

```
epoch =   2
Filtered State  Corrns        Filtered State cofactor matrix Qxx
   8289.594     -5.406        1.009225 -0.797965  0.033097 -0.026169
   6521.882     22.490       -0.797965  1.439797 -0.026169  0.047217
      6.823     -0.177        0.033097 -0.026169  0.506780 -0.000858
      3.738      0.738       -0.026169  0.047217 -0.000858  0.507243

epoch =   3
Filtered State  Corrns        Filtered State cofactor matrix Qxx
   8705.780      6.823        0.926924 -0.643621  0.030398 -0.021105
   6727.944    -18.189       -0.643621  1.218371 -0.021105  0.039955
      7.046      0.224        0.030398 -0.021105  0.494715 -0.004015
      3.141     -0.596       -0.021105  0.039955 -0.004015  0.496822

:
:

epoch =  19
Filtered State  Corrns        Filtered State cofactor matrix Qxx
  15366.544    -11.869        0.732074  0.288579  0.024112  0.009501
   9973.570     -6.708        0.288579  0.820826  0.009502  0.027033
      6.743     -0.391        0.024112  0.009502  0.364017  0.005200
      3.276     -0.221        0.009501  0.027033  0.005200  0.366470

epoch =  20
Filtered State  Corrns        Filtered State cofactor matrix Qxx
  15781.273     10.148        0.598119  0.158080  0.019704  0.005204
  10175.278      5.167        0.158080  0.847313  0.005203  0.027911
      7.077      0.334        0.019704  0.005203  0.358551  0.006055
      3.446      0.170        0.005204  0.027911  0.006055  0.361445
```

```
Filtered Values
Epoch Distance    Speed   Heading
      (metres)    (m/s)  (degrees)
 1       0.000    7.616   66.801
 2     461.400    7.779   61.286
 3     925.806    7.715   65.975
 4    1390.357    7.775   62.905
 5    1853.155    7.656   64.729
 6    2315.773    7.765   63.208
 7    2778.387    7.660   65.206
 8    3240.322    7.741   62.817
 9    3703.373    7.698   64.889
10    4165.683    7.713   63.805
11    4631.259    7.805   63.717
12    5091.795    7.550   64.711
13    5555.396    7.900   63.076
14    6020.782    7.618   64.761
15    6481.164    7.728   63.223
16    6945.300    7.744   64.584
17    7405.657    7.603   63.888
18    7872.215    7.945   63.889
19    8335.291    7.497   64.090
20    8796.471    7.872   64.039
```

The speed and heading of the ship at each epoch have been computed using equations (13) and (14) respectively. If the first two epochs are ignored, on the assumption that the starting values are not very close to the truth and the filter needs some time to "stabilise", then the average of the remaining values are 7.72 m/s and 64.08° respectively with ranges 0.4 m/s (epochs 18 and 19) and 3.1° (epochs 3 and 4). Bearing in mind that the data were generated for a ship travelling at 15 knots (equivalent to 7.72 m/s) on a heading of 064°, these are very close estimates of the "true" values. Also, the "true" position of the ship at the 20th epoch (using the initial coordinates and 15 knots on a heading of 064°) is 15781.691 E and 10175.743 N. The Kalman filter gives the ship's position as 15781.273 E and 10175.278 N (see the filtered state for epoch 20 in MATLAB output above), which is within 0.418 m and 0.465 m respectively of its "true" location.

The output from the MATLAB program also gives the cofactor matrix of the filtered state, i.e., the estimates of variances and covariances of the four elements of the state vector. The initial estimates for epoch t_1 (see step (g) above) were $s_E^2 = s_N^2 = 20\,\mathrm{m}^2$ and $s_{\dot{E}}^2 = s_{\dot{N}}^2 = 0.5\ \mathrm{m}^2/\mathrm{s}^2$ giving

$$\mathbf{Q}_{x_1} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

and for epoch t_2 (after one pass through the filter)

$$\mathbf{Q}_{x_2} = \begin{bmatrix} 1.009225 & -0.797965 & 0.033097 & -0.026169 \\ -0.797965 & 1.439797 & -0.026169 & 0.047217 \\ 0.033097 & -0.026169 & 0.506780 & -0.000858 \\ -0.026169 & 0.047217 & -0.000858 & 0.507243 \end{bmatrix}$$

and finally for epoch t_{20}

$$\mathbf{Q}_{x_{20}} = \begin{bmatrix} 0.598119 & 0.158080 & 0.019704 & 0.005204 \\ 0.158080 & 0.847313 & 0.005203 & 0.027911 \\ 0.019704 & 0.005203 & 0.358551 & 0.006055 \\ 0.005204 & 0.027911 & 0.006055 & 0.361445 \end{bmatrix}$$

Here the variance estimates are $s_E^2 = 0.598119\,\mathrm{m}^2$, $s_N^2 = 0.847313\,\mathrm{m}^2$ and $s_{\dot{E}}^2 = 0.358551\ \mathrm{m}^2/\mathrm{s}^2$, $s_{\dot{N}}^2 = 0.361445\ \mathrm{m}^2/\mathrm{s}^2$. These equate to standard deviations in position of 0.78 m E and 0.92 m N and standard deviations in velocity of 0.60 m/s E and 0.60 m/s N. Using equation (13) and the law of propagation of variances gives the estimated standard deviation of the ship's speed as 0.60 m/s.

We can see from this very limited analysis that the Kalman filter gives quite reasonable estimates of the position <u>and velocity</u> of the ship from a sequence of noisy distance measurements. This is a feature of a Kalman filter: you obtain information about the dynamics of your measurement platform.

CONCLUSION

Kalman filtering is an extension of the least squares technique as formulated by C.F. Gauss. Indeed, Krakiwsky (1975) shows that the Kalman filter equations can be derived from the basic least squares principle (minimizing sums of weighted squares of measurement and model residuals) used in deriving conventional methods of least squares adjustment of survey data. In this paper we have stated the filter equations and the primary and dynamic models upon which they are based and then used two examples to demonstrate their use. The first example, the determination of a distance by an EDM, is an attempt to show how a Kalman filter could be used to obtain a single estimate from a continuous sequence of measurements. In this case the system (the EDM) is in fact static but appears to be moving due to measurement errors.

The Kalman filter processes the data sequence providing a "best estimate" at each measurement epoch based on all the previous measurements and their measurement precisions. The second example is a more conventional navigation problem: and one that has been used in a number of texts (e.g., Cross 1992) to demonstrate the usefulness of the Kalman filter. In this example the measurements are non-linear functions of the elements of the state vector and we have shown how a linearized primary measurement model is obtained and used. We have also shown, by assuming that position is a continuous function of time, how the members of the dynamic model are obtained – the transition matrix **T**, the coefficient matrix **H** and the system driving noise **w** where the dynamic model corrections are $\mathbf{v}_m = \mathbf{H}\mathbf{w}$. The assumption that position is a continuous function of time may not be correct in all navigation problems, but is adequate in our example.

In both of our examples, data have been generated to simulate actual measurements affected by random errors. These simulated measurements have normally distributed random errors and the measurements are independent, i.e., there is no correlation[3] between measurements at different epochs. This may not reflect actual measurements obtained from real situations, where there could be some unaccounted-for systematic errors and/or possibly unknown correlations between measurements. In such cases, careful analysis of residuals and data may be required to identify systematic errors (or deficiencies in the models) and correlated data; with a possible need to modify the Kalman filter equations to allow for correlation. Bearing in mind this assumption, our examples demonstrate how closely the Kalman filter estimates approach the "true" values, even with a limited amount of data (the second example) and quite noisy measurements.

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[3] Correlation is a statistical measure of the linear independence of measurements. If two measurements are independent then their correlation will be zero. Correlated measurements require special treatment in any least squares process.